Announcements

1) Career Fair

Thursday, 11–2,

Fairlane Center North

2) HW and extra credit:

HW #7 and basis extra credit due Friday, bonus from Exam 2 due Monday

Note: (Characteristic polynomial)

If A is an $n \times n$ matrix, the characteristic polynomial $p(x)$ is given by $\det(A - xI_n)$.

(We usually use λ instead of x)

Definition: (similarity)

Two $n \times n$ matrices A and B are similar if there exists an invertible $n \times n$ matrix S with

$$\boxed{S^{-1}AS = B},$$

or equivalently,

$$\boxed{A = SBS^{-1}}$$

What do we want?

Given A, we want A similar to a diagonal matrix.

How do we get it?

Eigenvalues and eigenvectors for A!

Example 1: (3×3)

Let $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

Is A similar to a diagonal matrix? If so, find the matrix and the similarity matrix S.

Step 1: Calculate the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of A. Note! you may have less than three, but never more than 3.

We want $\det(A - \lambda I_3) = 0$.

Or just let Wolfram Alpha find the eigenvalues...

We get

$$\lambda_1 = 7$$

$$\lambda_2 = \frac{1}{2}(5+3\sqrt{5})$$

$$\lambda_3 = \frac{1}{2}(5-3\sqrt{5})$$

Diagonal matrix:

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & \frac{1}{2}(5+3\sqrt{5}) & 0 \\ 0 & 0 & \frac{1}{2}(5-3\sqrt{5}) \end{bmatrix}$$

<u>Step 2</u>: Find the eigenvectors associated to $\lambda_1, \lambda_2, \lambda_3$.

If $A v_1 = \lambda_1 v_1$

$A v_2 = \lambda_2 v_2$

$A v_3 = \lambda_3 v_3$

for v_1, v_2, v_3 nonzero vectors,

then

$$S = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \text{ if}$$

$S^{-1} A S =$ diagonal.

We want, with $\lambda_1 = 7$,

a vector v_1 with

$$A v_1 = 7 v_1 .$$

Same as

$$(A - 7 I_3) v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So you need to find

$$\text{Nul}(A - 7 I_3).$$

$$\text{rref}\left(A - 7 I_3 \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right)$$

$$\left[\begin{array}{c|c} A-7I_3 & \begin{matrix}0\\0\\0\end{matrix} \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -6 & 3 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{rref} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So if $v_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then

$x_1 = 0$, $x_2 = 0$, x_3 is arbitrary.

Let's choose $x_3 = 1$.

(can't choose $x_3 = 0$)

We get $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

is an eigenvector for

the eigenvalue 7.

We then must also row reduce

$$\left[\begin{array}{c|c} A-\left(\frac{1}{2}(5+3\sqrt{5})\right)I_3 & \begin{matrix}0\\0\\0\end{matrix}\end{array}\right]$$

and

$$\left[\begin{array}{c|c} A-\left(\frac{1}{2}(5-3\sqrt{5})\right)I_3 & \begin{matrix}0\\0\\0\end{matrix}\end{array}\right]$$

—or— use Wolfram Alpha's "eigenvalue" command— or just enter the matrix!

We could take

$$v_2 = \begin{bmatrix} \frac{1}{2}(-1+\sqrt{5}) \\ 1 \\ 0 \end{bmatrix}$$

and

$$v_3 = \begin{bmatrix} \frac{1}{2}(-1-\sqrt{5}) \\ 1 \\ 0 \end{bmatrix}.$$

Then $S = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ ✓

$$= \begin{bmatrix} 0 & \frac{1}{2}(-1+\sqrt{5}) & \frac{1}{2}(-1-\sqrt{5}) \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Note: If you are asked to find more than one diagonal matrix D and similarity S with

$$S^{-1} A S = D,$$

then you permute the diagonal of D and permute the columns of S in exactly the same way.

So for the previous example, another diagonal is given by

$$\begin{bmatrix} \frac{1}{2}(5+3\sqrt{5}) & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & \frac{1}{2}(5-3\sqrt{5}) \end{bmatrix}$$

(flipped 1st two diagonal entries)

Flip the first two columns of the old S to get the new similarity.

Example 2: (powers of matrices)

Find a general formula for A^n if n is a natural (counting) number and $A = \begin{bmatrix} 5 & 2 \\ 2 & -6 \end{bmatrix}$.

Note: this means a formula for $A, A^2, A^3, A^4, A^5, \ldots$

Using wolfram alpha,

the eigenvalues are

$$\frac{1}{2}\left(-1 \pm \sqrt{137}\right)$$

with eigenvectors

$$\begin{bmatrix} \frac{1}{4}\left(11 - \sqrt{137}\right) \\ 1 \end{bmatrix} \quad (+)$$

$$\begin{bmatrix} \frac{1}{4}\left(11 + \sqrt{137}\right) \\ 1 \end{bmatrix} \quad (-)$$

With

$$D = \begin{bmatrix} \frac{1}{2}(-1+\sqrt{137}) & 0 \\ 0 & \frac{1}{2}(-1-\sqrt{137}) \end{bmatrix}$$

and

$$S = \begin{bmatrix} \frac{1}{4}(11-\sqrt{137}) & \frac{1}{4}(11+\sqrt{137}) \\ 1 & 1 \end{bmatrix}$$

We have $S^{-1}AS = D$.

Then $A = SDS^{-1}$.

$$A^2 = (SDS^{-1})(SDS^{-1})$$

$$= SD\underbrace{(S^{-1}S)}_{I_2}DS^{-1}$$

$$= SD^2S^{-1}.$$

Similarly, $A^3 = SD^3S^{-1}$

and $A^n = SD^nS^{-1}$

$$A^n = S\begin{bmatrix} \left(\frac{1}{2}(-1+\sqrt{137})\right)^n & 0 \\ 0 & \left(\frac{1}{2}(-1-\sqrt{137})\right)^n \end{bmatrix}S^{-1}$$